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A NEW APPROACH FOR THE LOGARITHMS OF REAL NEGATIVE NUMBERS

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ABSTRACT

Logarithms were defined for real positive numbers with a real positive base, but were later extended to real negative numbers with a real positive base. Logarithms of real negative numbers to a real positive base were defined as complex numbers. The Neparian logarithms take into consideration the hyperbola transcribed by the function $f(x) = 1/x$, $x>0$ for real positive axis. On a parallel analogy, we extend this concept to the real negative axis for the hyperbola transcribed by $1/x$ for $x<0$. This paper examines the concept of logarithms from its basics to prove that the logarithms of real negative numbers to real negative base are real numbers. The concept of logarithms as applicable to both real positive and real negative numbers has been generalized.

Keywords: Logarithms; real negative numbers; exponential function, computational sciences.

INTRODUCTION

It is a known fact that in the field of mathematics, logarithms have introduced new transcendent sizes and have widened the field of comprehension of numericals. The origin of the concept of logarithms goes back to John Napier (1950-1617) who used arithmetic and geometric progressions as a basis of his work. Neparian logarithms as given by Napier are shifted by $10^7\ln 10^7$ with respect to the logarithms used presently (Lexa, 2013). Napier designed an ingenious mathematical tool at a time when the concepts of modern mathematics were not laid. The Napier's concept evolved into the present day logarithmic function which became indispensable to the development of science and technology.

1 Development of the concept

Napier, initially described logarithms through geometry and not as the inverse of the exponential function. Today, the natural logarithmic function is usually defined as the inverse of the natural exponential function or through the integral equation

$$\ln(x) = \int_1^x (1/t)dt, x > 0 \tag{1.1}$$

Logarithms were initially defined for all real positive numbers, and it was Euler who discovered ways to describe logarithms in terms of power series, and also successfully defined logarithms of complex and negative numbers, which expanded the field of logarithms. Earlier to Euler, logarithms of real negative numbers were defined as equal to logarithms of real positive numbers of the same magnitude (Wikipedia, 2013) as given by

$$\log_b(x) = \log_b(-x)$$

This fact emanates from the additivity property of logarithms, given by

$$2\log_b(-x) = \log_b(-x)^2 = \log_b(x)^2 = 2\log_b(x) \tag{1.2}$$

Later Euler gave a formula known as Euler's identity, given by

$$e^{i\pi} + 1 = 0, \tag{1.3}$$

where '*e*' *is Euler's constant and* $i = \sqrt{-1}$. This defined the logarithm of (-1):

$$\log_e(-1) = i\pi .$$

This paved a way for the field of logarithms of complex numbers.

Logarithmic functions play a significant role in solving complicated mathematical expressions and thus are important in the field of mathematics. Shannon (1948) whose discovery of entropy made a remarkable impact for measuring uncertainty contained in probabilistic experiments, had to develop his mathematical model involving logarithmic function for measuring information content. Many other researchers extended Shannon's (1948) entropy in terms of logarithmic functions under diverse situations. Recently, Parkash and Thukral (2010) investigated and developed certain logarithmic models for measuring diversity in biological systems.

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Logarithms are extensively used in various disciplines like population dynamics Vandermeer (2010), social sciences (2013) and other fields. Developments in the field of logarithms may be attributed to several authors, notably Joost Burgi, Johannes Kepler, Gregoire de Saint-Vincent, Leonard Euler and others (Lexa, 2013; Wikipedia, 2013). Euler first defined the exponential function and related it to the natural logarithm. The present day terminology of the logarithms is attributed to Leibnitz and Newton (Lefort, 2013).

The present paper provides a new approach to define the logarithms of negative real numbers using a multiplicative constant (-1) as a coefficient to the logarithm base. The necessity for the present work arises to fulfill the gaps in the derivation of the logarithmic functions for the negative real numbers defined over the negative real axis. The software used in this paper are
http://rechneronline.de/function-graphs/,
http://wims.unice.fr and MS-Excel software.

As is seen from figure 1, $y = \frac{1}{x}$ transcribes hyperbolas in the first and the third quadrants. The curve in the first quadrant defines the logarithmic function for positive real numbers whereas the curve in the third quadrant has never been considered for defining any such function. The curve for $y = \ln(x)$ for positive real x-axis is given in figure 2. We stress here that a similar curve can be constructed for the real negative axis using the graphical presentation provided in the third quadrant of figure 2.

2 Logarithms of real negative numbers

In the literature, the logarithms of real positive numbers are defined as the inverse function of exponential function, given by

$$b^y = x \tag{2.1}$$

such that logarithm of x to the base b is defined as,

$$\log_b x = y \tag{2.2}$$

where x is a real positive number, y is a real number, and b is a real positive number base of the logarithm (generally 10, or 2 or e). Multiply equation (2.1) with a constant $c = \pm 1$, referred to herein as the coefficient,

$$c(b^y) = cx. \tag{2.3}$$

Keeping in view the equation (2.3), we now define the logarithmic function as

$$\log_{c,b}(cx) = y \tag{2.4}$$

where c represents the coefficient ($\pm$ 1) to the base b>1 and x>0.

For c = +1, the logarithmic equation (2.2) for real positive numbers is obtained. For, c = -1 we get, the logarithms of real negative numbers as

$-(b^y) = -x$, b>1 and x>0, i.e.,

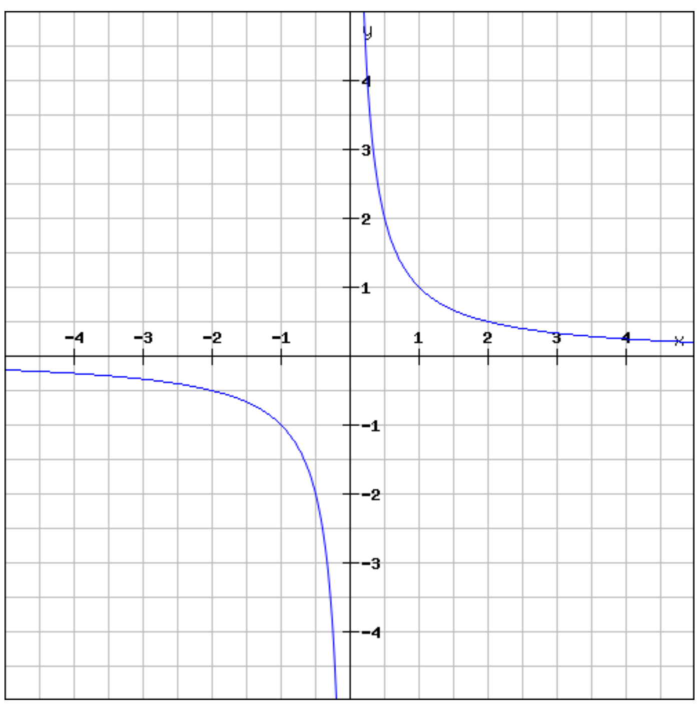

Fig. 1. Graph between x and $1/x$.

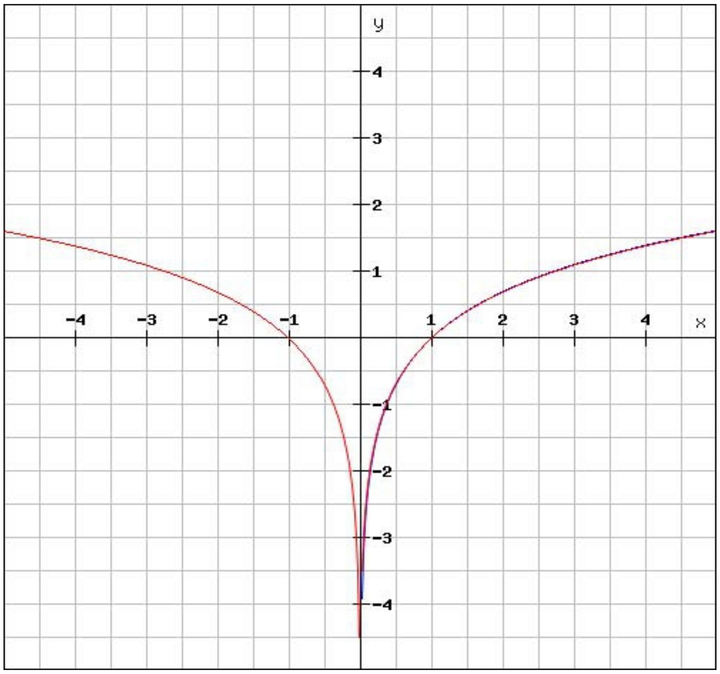

Fig. 2. Graph between *x* and lnn(*x*) (left), and ln(*x*) (right)

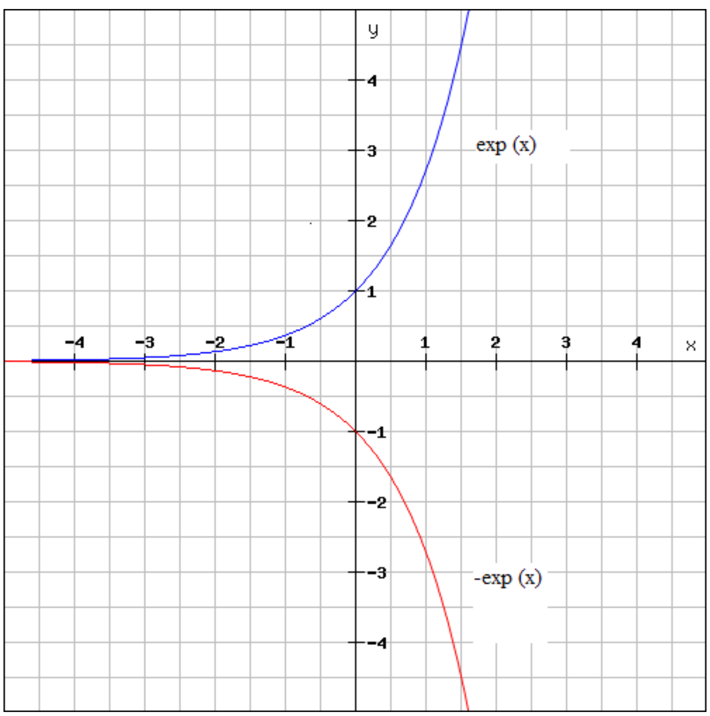

Fig. 3. Graph between *x* and exp(*x*) (upper curve), and -exp(*x*) (lower curve)

$$\log_{-1,b}(-x) = y \qquad (2.5)$$

For example,

$\log_{-1,10}(-100) = 2$, since, $-(10^2) = -100$.

In this case, the coefficient (-1) is not raised to the power '*y*' and is a multiplicative constant only. In the terminology of the present context, the base of a logarithm is *cb*, where *b*>1, and *c* = 1 for logarithms on the positive real axis '*x*', and *c* = -1 for the negative real axis '*x*'.

3. Necessity for the present concept

Under the notion held till now, the logarithm of a real negative number say, x = -100, to the base 'b', say, b=10, is $\log_{10}(-100) = y$, such that,

$$b^y = -x. \tag{3.1}$$

That is, the logarithm of (– 100) will be the value of y that satisfies the equation,

$$(10)^y = -100. \tag{3.2}$$

Equation (3.2) contradicts equation (2.5) since the coefficient, c = -1 multiplies only on one side of the equation (2.1). Another alternative to evaluate this logarithm would be

$$(-b)^y = -x. \tag{3.3}$$

That is, the logarithm of - 100 will be the value of y which satisfies the equation

$$(-10)^y = -100. \tag{3.4}$$

Equation (3.4) contradicts equation (2.5) since the coefficient (-1) on the left hand side is not to be raised to the power y. The logarithms of real positive or real negative numbers should therefore be defined as per the equations (2.3, 2.4).

Thus, to obtain the logarithm of (-100), we apply equation (2.5) as follows:

$$\log_{-1,10}(-100) = 2, \tag{3.5}$$

Since

$$-(10)^2 = -100 \tag{3.6}$$

which justifies the proposed method to find the logarithm of negative real numbers. If the base of the function is Euler's constant e, then

$$-(e^y) = -x, \tag{3.7}$$

that is,

$$\log_{-1,e}(-x) = y. \tag{3.8}$$

It is self evident that logarithm of a real positive number with a coefficient of c = 1, is equal to the logarithm of real negative number with a coefficient of c = -1. The logarithm of -1 to the base –e will be,

$\log_{-1,e}(-1) = 0$, Since,

$$-(e^0) = -1. \tag{3.9}$$

The inverse of the natural negative logarithmic function is, $-(e^x)$. The present paper also proves the existence of negative exponential function. A graph for positive and negative exponential functions is given in figure 3.

For convenience, log to the base (-e) may be called lnn(x) (log natural negative). In its integral form lnn (x) may be defined as

$$\log_{-1,e}(-x) = \int_{-1}^{-x} \frac{1}{t} dt = -\int_{-x}^{-1} \frac{1}{t} dt, \text{ for } x>0. \tag{3.10}$$

The properties of logarithms of negative real numbers to a negative base are given in table 1.

Table 1. Properties of logarithms of real negative and real positive numbers as per the proposed concept.

Property	Coefficient	Explanation
General definition of logarithm	c = +1, or c = -1	$\log_{c,b}(cx) = y$ Such that $c(b^y) = cx,\ x > 0, b > 1$
	c = 1	$\log_{+1,b}(x) = y$, or $\log_b(x) = y$ Such that $b^y = x, \quad x > 0, b > 1$
	c = -1	$\log_{-1,b}(-x) = y$ Such that $-(b^y) = -x \quad x > 0, b > 1$

Table 1 continue..

Property	Coefficient	Explanation
Earlier definition of logarithms of negative real numbers with a positive real base	c_1 = +1, c_2 = -1	$\log_{c_1,b}(c_2 x) = y$ Such that $c_1(b^y) = c_2 x$, $x > 0, b > 1$ Or, $\log_{+1,b}(-x) = y$ Such that $(b^y) = -x$, $x > 0, b > 1$, y is a complex number.
Equivalence		$\log_{-1,b}(-x) = \log_{+1,b}(+x)$, $x > 0, b > 1$
Product	c_1 and c_2 are same or different real number coefficients $(c_1, c_2 = \pm 1)$.	$\log_{c_1,b}(c_1 x_1) + \log_{c_2,b}(c_2 x_2) = \log_{c_1 c_2,b}(c_1 c_2 x_1 x_2)$, $x_1, x_2 > 0, b > 1$
	c_1 , c_2 = +1	$\log_{+1,b}(+x_1) + \log_{+1,b}(+x_2) = \log_{+1,b}(x_1 x_2)$, $x_1, x_2 > 0, b > 1$
	c_1 , c_2 = -1	$\log_{-1,b}(-x_1) + \log_{-1,b}(-x_2) = \log_{+1,b}(x_1 x_2)$, $x_1, x_2 > 0, b > 1$
	c_1 = -1, c_2 = +1 (or *vice versa*)	$\log_{-1,b}(-x_1) + \log_{+1,b}(x_2) = \log_{-1,b}(-x_1 x_2)$, $x_1, x_2 > 0, b > 1$
Quotient	When c_1 , c_2 are the same or different real number coefficients c_1 , $c_2 = \pm 1$.	$\log_{c_1,b}(c_1 x_1) - \log_{c_2,b}(c_2 x_2) = \log_{(c_1/c_2),b}(c_1 x_1 / c_2 x_2)$, $x_1, x_2 > 0, b > 1$
	When the coefficients of both real numbers x_1 and x_2 are negative (c_1 , c_2 = -1).	$\log_{-1,b}(-x_1) - \log_{-1,b}(-x_2) = \log_{+1,b}(x_1 / x_2)$, $x_1, x_2 > 0, b > 1$
	When the coefficient of real number x_1 is negative -1 and x_2 is positive +1.	$\log_{-1,b}(-x_1) - \log_{+1,b}(x_2) = \log_{-1,b}(-x_1 / x_2)$, $x_1, x_2 > 0, b > 1$
Power, $c(x^n)$	c = +1, or c = -1	$\log_{c,b} c(x^n) = n(\log_{c,b} cx)$, $x > 0, b > 1$
	c = +1	$\log_{+1,b}[+(x^n)] = n[\log_{+1,b}(x)]$, $x > 0, b > 1$, The log will be a real number.
	c = -1	$\log_{-1,b}[-(x^n)] = n[\log_{-1,b}(-x)]$, $x > 0, b > 1$, The log will be a real number.
Power, $(-x)^n$		$\log_b(-x)^n = n[\log_b(-x)]$, $x > 0, b > 1$ The value of log will be a complex number.
Base switch	c = +1, or c = -1	$\log_{c,b}(cx) = \dfrac{1}{\log_{c,x}(cb)}$, $x > 0, b > 1$
	c = +1	$\log_{+1,b}(+x) = \dfrac{1}{\log_{+1,x}(+b)}$, $x > 0, b > 1$

continue…

Table 1 continue..

Property	Coefficient	Explanation
Base switch		
	c = -1	$\log_{-1,b}(-x) = \dfrac{1}{\log_{-1,x}(-b)}$, $x > 0, b > 1$
Base change	*c* = +1, or *c* = -1	$\log_{c,b}(cx) = \dfrac{\log_{c,e}(cx)}{\log_{c,e}(cb)}$, $x > 0, b > 1$
	c = +1	$\log_{+1,b}(+x) = \dfrac{\log_{+1,e}(+x)}{\log_{+1,e}(+b)}$, $x > 0, b > 1$
	c = -1	$\log_{-1,b}(-x) = \dfrac{\log_{-1,e}(-x)}{\log_{-1,e}(-b)}$, $x > 0, b > 1$
Antilog of log	*c* = +1, or *c* = -1	If $\log_{c,b}(cx) = y$, then $x = b^y$, $x > 0, b > 1$
	c = +1	If $\log_{+1,b}(+x) = y$, then $x = b^y$
	c = -1	If $\log_{-1,b}(-x) = y$, then $-x = -b^y$
Inverse functions of log natural and log natural negative	*c* = +1, or *c* = -1	ce^x, $x > 0$
	c = +1	e^x, $x > 0$
	c = -1	$-e^x$, $x > 0$
Log natural of coefficient, *c*	*c* = +1, or *c* = -1	$\log_{c,e} c = 0$
	c = +1	$\log_{+1,e}(+1) = 0$
	c = -1	$\log_{-1,e}(-1) = 0$
Integral form	*c* = +1, or *c* = -1	$\log_{c,e}(cx) = \int_{c}^{cx} (1/t)dt$, $x>0$
	c = +1	$\log_{+1,e}(+x) = \int_{1}^{x} (1/t)dt$, $x>0$.
	c = -1	$\log_{-1,e}(-x) = \int_{-1}^{-x} (1/t)dt = -\int_{-x}^{-1} (1/t)dt$, $x>0$.

CONCLUSION

The present concept uses hyperbola in quadrant III of the graph for $1/x$, and proves that the logarithms of real negative numbers on a real negative base are real numbers. The concept will have wide applications in basic and computational sciences.

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